Ranking Opinions with Few States in Population Protocols

Joint work with Tom-Lukas Breitkopf, Julien Dallot, and Stefan Schmid

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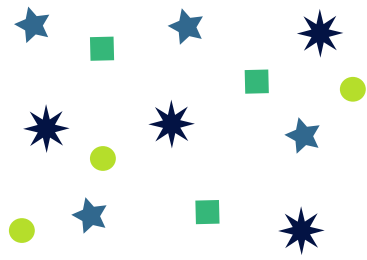
PODC 2026

Population Protocols

- n agents, each with an initial state
- At each time step, two agents chosen by a scheduler meet
- They see each other's state, and update theirs according to some rule

Example: Agent with more angles gets duplicated



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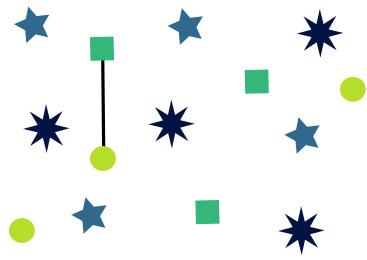

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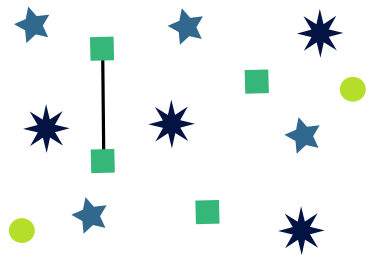

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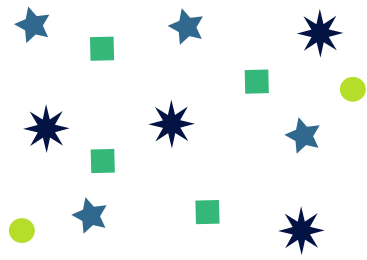

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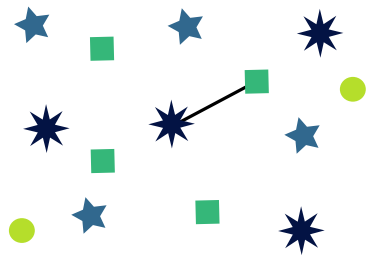

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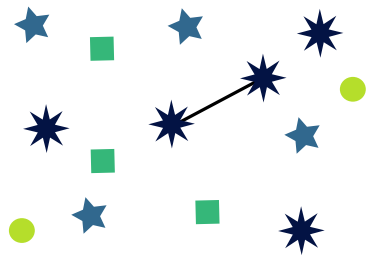

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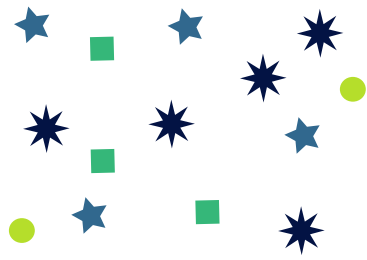

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The Plurality Consensus Problem

- Input: an element in $[k]$
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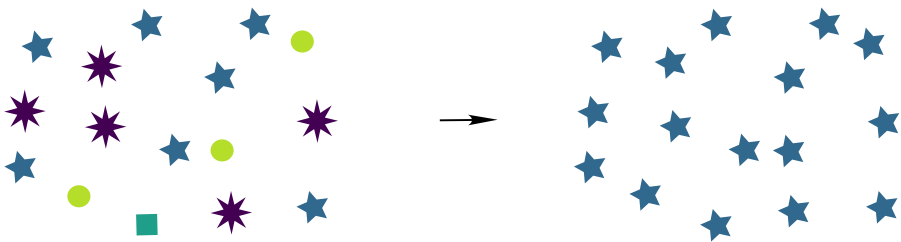

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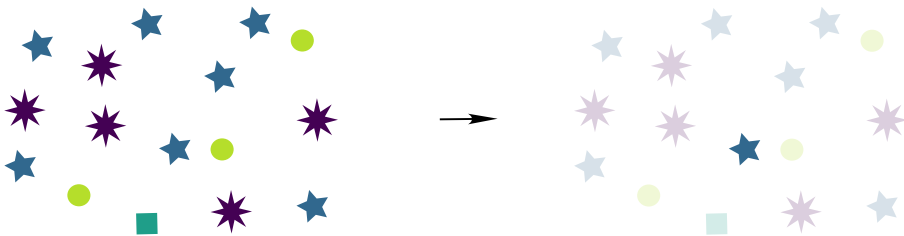

The CIRCLES protocol: Low-Space with a weakly fair scheduler

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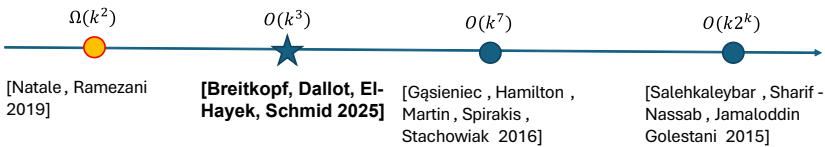

Comparison with the Binary Consensus Problem

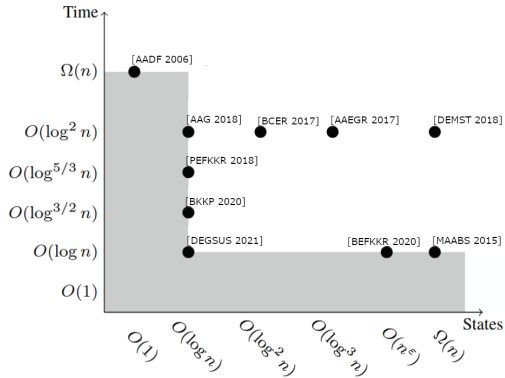

Figure 2: State of the art of the **Binary** consensus problem. Figure by [DEGSUS, FOCS'21]

Stable Matchings/Hetero Marriages?

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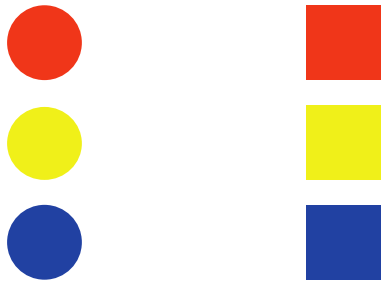

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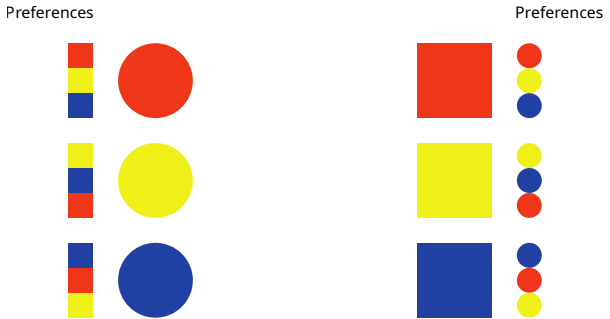

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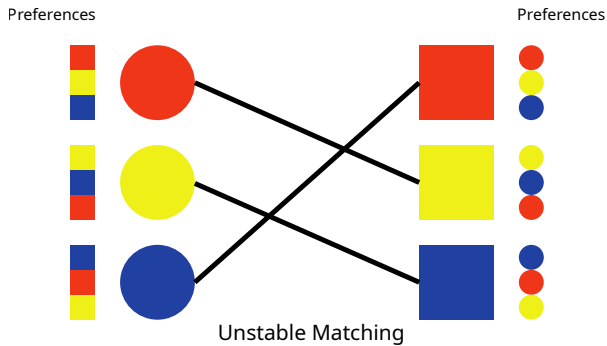

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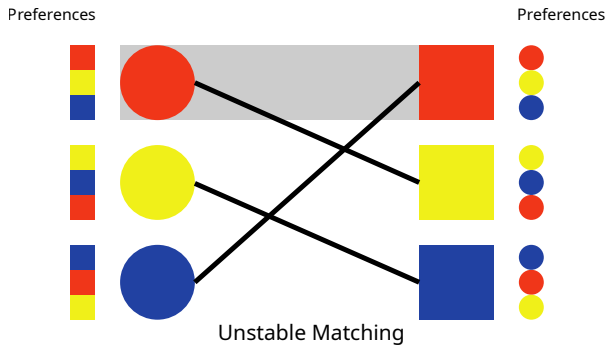

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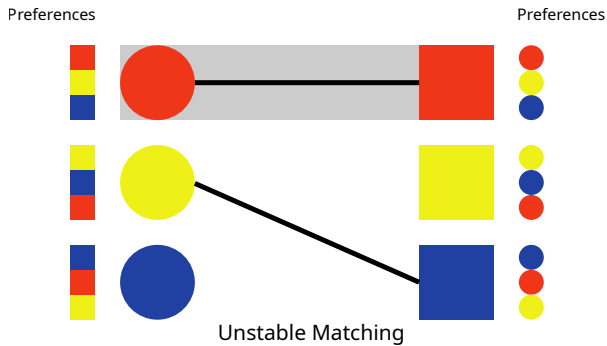

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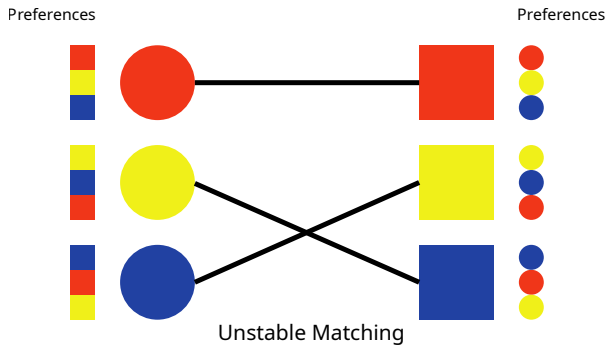

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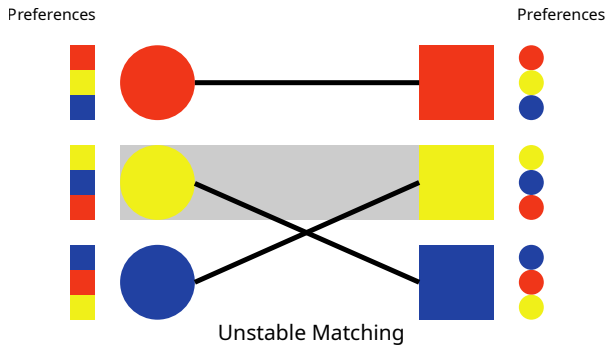

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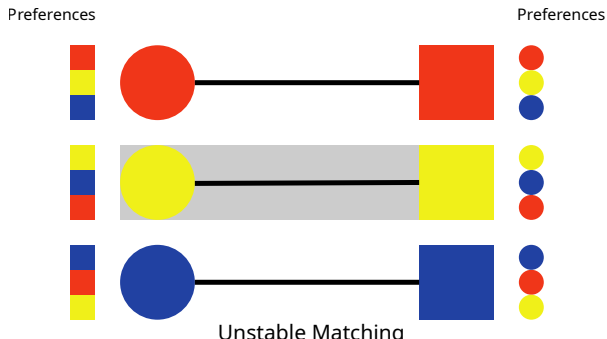

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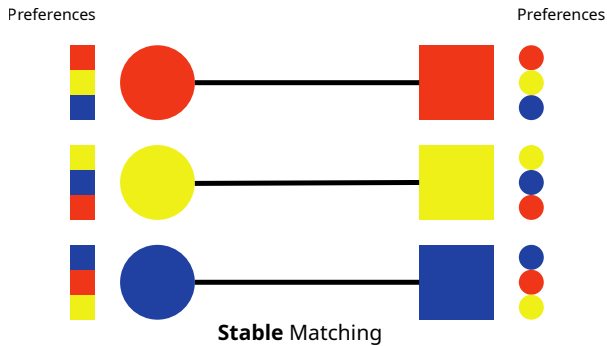

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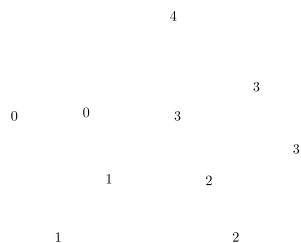

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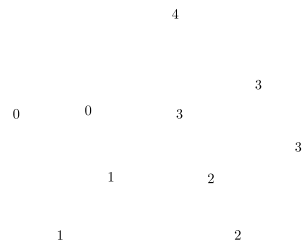

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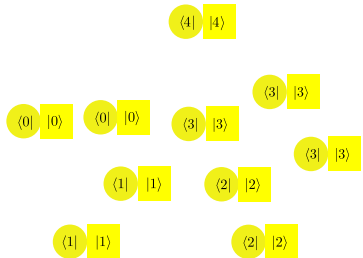

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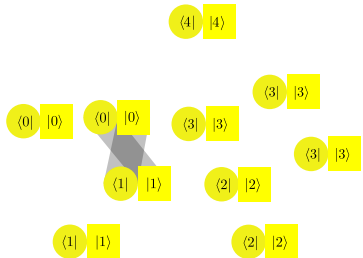

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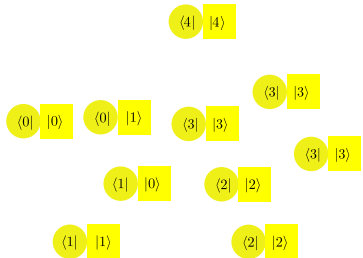

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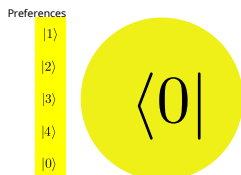

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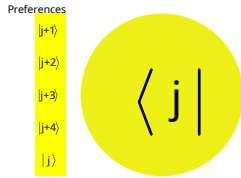

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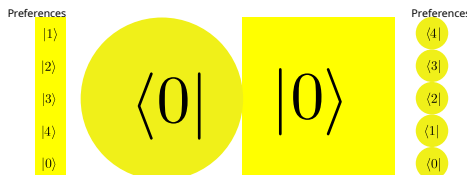

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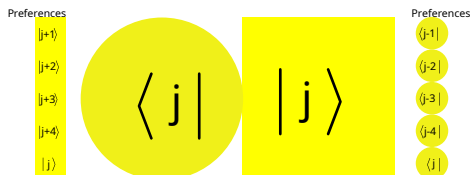

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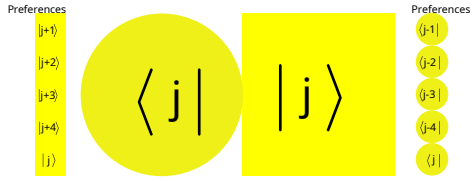

- Output function: $\mathcal{O}(\langle i|j\rangle) = \begin{cases} i & \text{if } i = j \\ \perp & \text{otherwise} \end{cases}$

A small example

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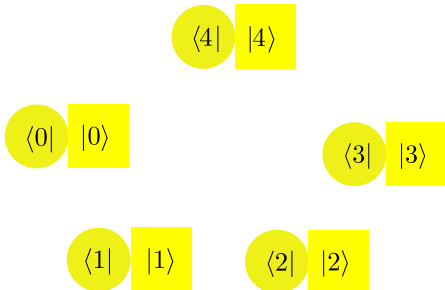

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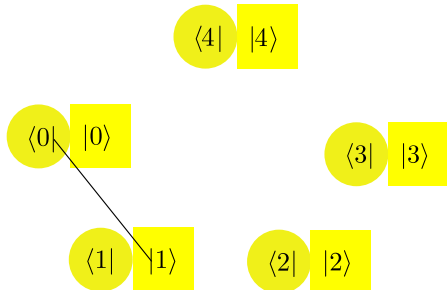

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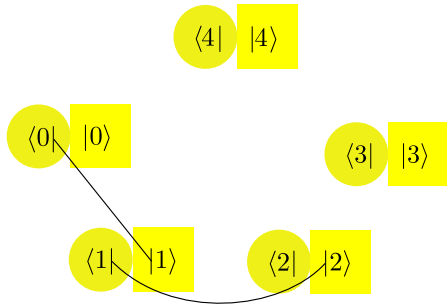

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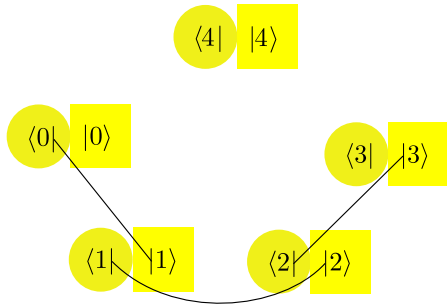

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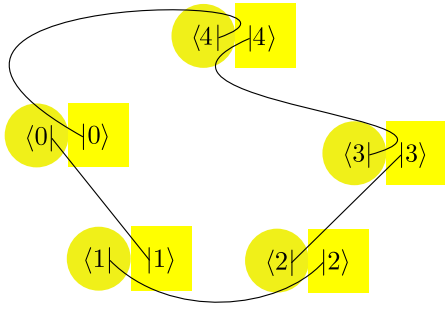

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- We also give an *ordinal* potential function to show that this configuration is reachable.

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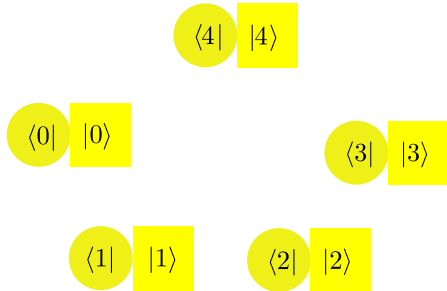

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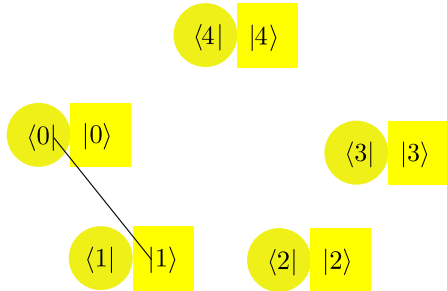

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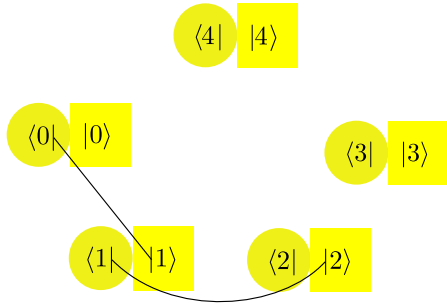

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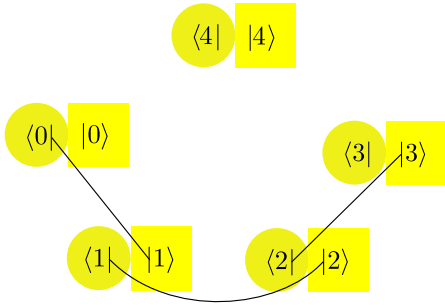

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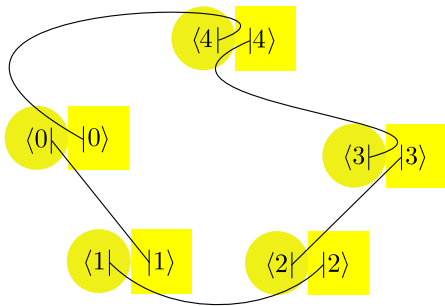

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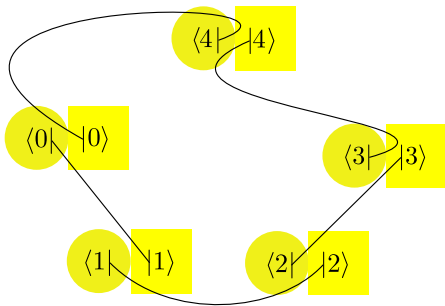

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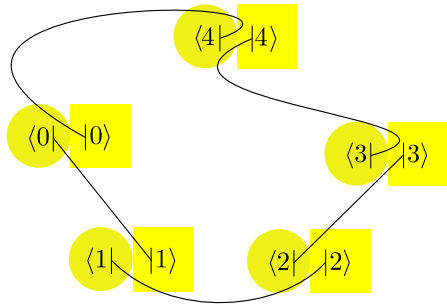
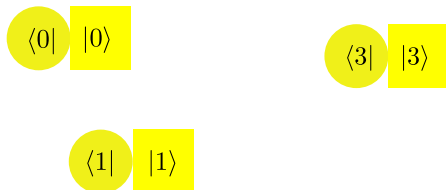


- This is a *Circle*!
- This is now set in stone: no interaction will change any of these combinations.

Finding *a* stable configuration

- What if the circle is incomplete?



Finding *a* stable configuration

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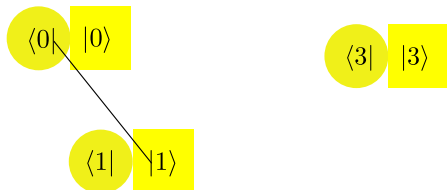

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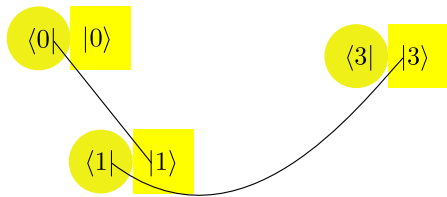

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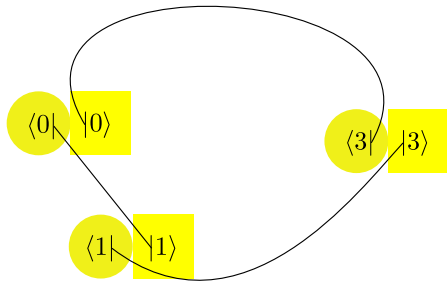
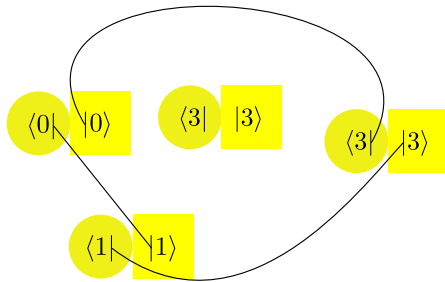


- This is *also* a Circle

Finding *a* stable configuration

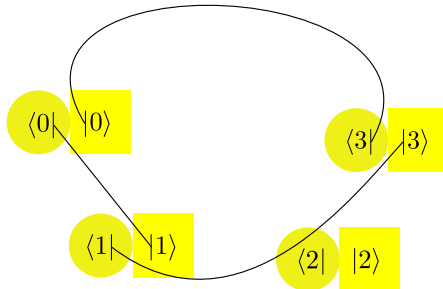
- What if the circle is incomplete?



- This is *also* a Circle
- Adding an element that is already in the Circle does not affect it

Finding *a* stable configuration

- What if the circle is incomplete?



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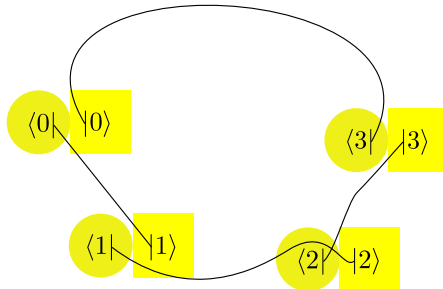

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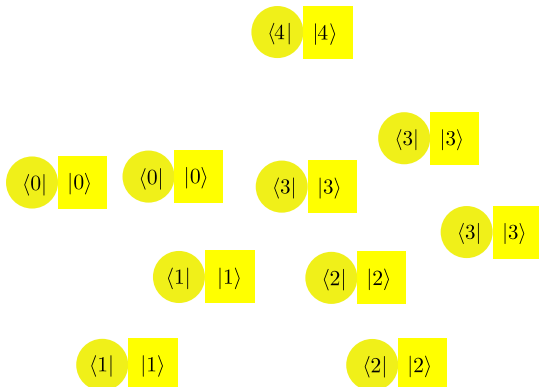
Finding *a* stable configuration

- Let us add some complexity



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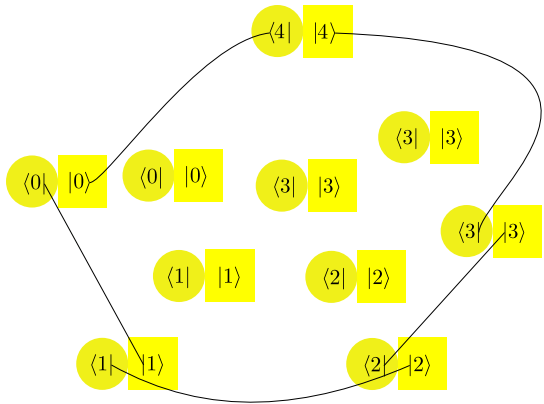

- We start with the biggest possible circle

Finding *a* stable configuration

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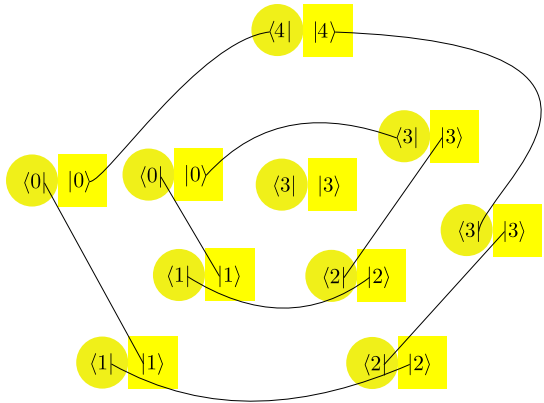

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Finding *a* stable configuration

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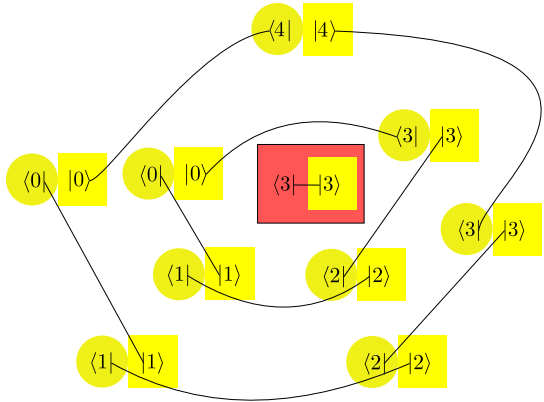
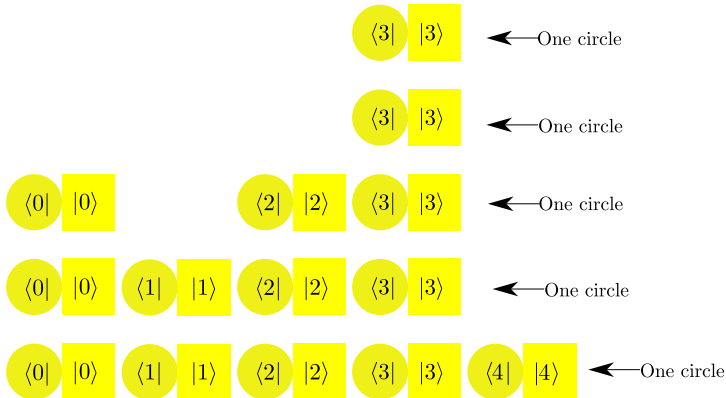


- We start with the biggest possible circle, then the next
- The final circle has one element in it: the majority element

Finding *a* stable configuration

- Switching to a bar chart view



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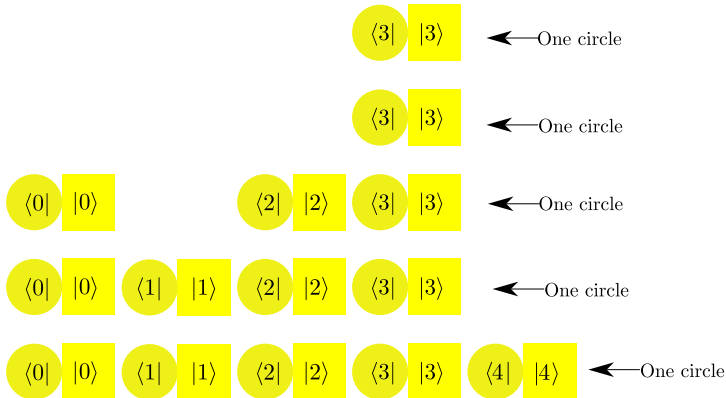

- We have as many circles as the majority element

Is this configuration unique?

- *weight* function: $w(\langle i| \, , |j\rangle) = \begin{cases} k & \text{if } i = j \\ (j - i) \bmod k & \text{otherwise (a number beween } 1 \text{ and } k - 1) \end{cases}$

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- The potential function argument also ensures we do not end up in an infinite loop.

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